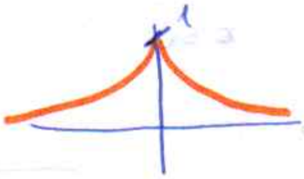
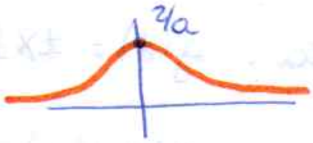
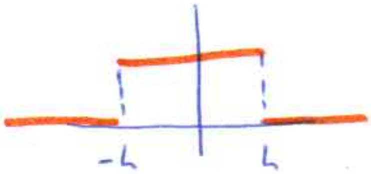
Transformadas calculadas.

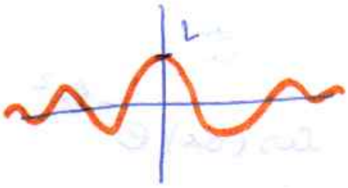
A) $f(x) = e^{-a|x|}$, $a>0$ $\qquad$ $\hat{f}(\omega) = \dfrac{2a}{a^2+\omega^2}$



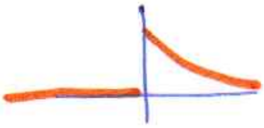


B) $f(x) = \begin{cases} 1 & |x|<L \\ 0 & |x|\geq L \end{cases}$ $\qquad$ $\hat{f}(\omega) = \dfrac{2\,\mathrm{sen}(\omega L)}{\omega}$

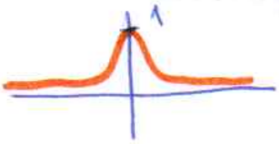


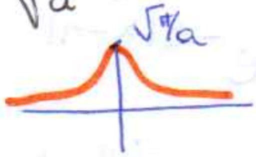


D) $f(x) = \begin{cases} e^{-x} & x>0 \\ 0 & x\leq 0 \end{cases}$ $\qquad$ $\hat{f}(\omega) = \dfrac{1}{1+i\omega} = \dfrac{1}{1+\omega^2} - i\,\dfrac{\omega}{1+\omega^2}$

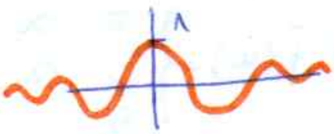


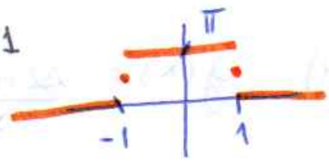
E) $f(x) = e^{-ax^2}$ $\qquad$ $\hat{f}(\omega) = \sqrt{\frac{\pi}{a}}\; e^{-\omega^2/4a}$



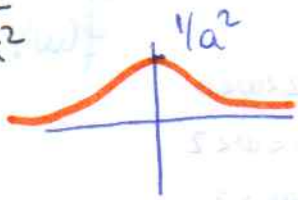


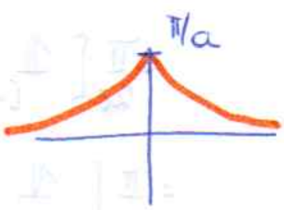
$f(x) = \dfrac{\mathrm{sen}(x)}{x}$ $\qquad$ $\hat{f}(\omega) = \begin{cases} 0 & |\omega|>1 \\ \pi & |\omega|<1 \\ \pi/2 & |\omega|=1 \end{cases}$





$f(x) = \dfrac{1}{a^2+x^2}$ $\qquad$ $\hat{f}(\omega) = \dfrac{\pi}{a}\, e^{-a|\omega|}$





Cálculo de transformadas usando propiedades.

1) $g(x) = \begin{cases} x \cdot e^{-x} & \text{si } x>0 \\ 0 & \text{si } x \leq 0 \end{cases}$

Resulta: $g(x) = x f(x)$, con $f(x) = \begin{cases} e^{-x} & x>0 \\ 0 & x \leq 0 \end{cases}$

g y f son absolutamente integrables $\Rightarrow$

$$\hat{g}(w) = \widehat{xf}(\omega) = i \hat{f}'(\omega) = i \left(\frac{1}{1+i\omega} \right)' = i \frac{(-1)}{(1+i\omega)^2} \cdot i \boxed{= \frac{1}{(1+i\omega)^2}}$$

2). $h(x) = \cos(3x) e^{-4x^2}$

Como $h(x) = \cos(3x) \cdot f(x)$ con $f(x) = e^{-4x^2}$,

$$\hat{h}(\omega) = \frac{1}{2}\left[\hat{f}(\omega-3) + \hat{f}(\omega+3) \right] = \frac{1}{2}\left[\sqrt{\frac{\pi}{4}}\, e^{-\frac{(\omega-3)^2}{16}} + \sqrt{\frac{\pi}{4}}\, e^{-\frac{(\omega+3)^2}{16}} \right]$$

$$= \frac{\sqrt{\pi}}{4} \cdot \left[e^{-\frac{(\omega-3)^2}{16}} + e^{-\frac{(\omega+3)^2}{16}} \right]$$

3) $h(x) = e^{-5|x-1|} = f(x-1)$ siendo $f(x) = e^{-5|x|}$

$$\Rightarrow \hat{h}(\omega) = e^{-i\omega} \cdot \hat{f}(\omega) = e^{-i\omega} \cdot \frac{10}{25+\omega^2}$$

4) $g(x) = \frac{\operatorname{sen}^2 x}{x} = \operatorname{sen} x \cdot f(x)$ siendo $f(x) = \frac{\operatorname{sen} x}{x}$

$\hookrightarrow \hat{f}(\omega) = \begin{cases} \pi & \text{si } |\omega|<1 \\ 0 & \text{si } |\omega|>1 \\ \pi/2 & |\omega|=1 \end{cases}$

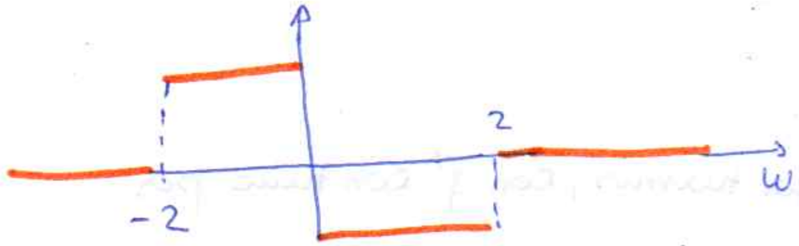
$$\Rightarrow \hat{g}(\omega) = \frac{1}{2i}\left[\hat{f}(\omega-1) - \hat{f}(\omega+1) \right]$$

$$= \frac{\pi}{2i}\left[\mathbb{1}_{[-1,1]}(\omega-1) - \mathbb{1}_{[-1,1]}(\omega+1) \right]$$

$\hat{f}(\omega) = \pi\, \underbrace{\mathbb{1}_{[-1,1]}(\omega)}$

función característica del intervalo $[-1,1]$.

$$= \frac{\pi}{2i}\left[\mathbb{1}_{[0,2]}(\omega) - \mathbb{1}_{[-2,0]}(\omega) \right] = \begin{cases} -\pi/2i & -2<\omega<0 \\ \pi/2i & 0<\omega<2 \\ 0 & |\omega|>2 \end{cases}$$



$\hookrightarrow$ Imag $(\hat{g}(w))$

5) $h(x) = \dfrac{5}{x^2+4x+8}$

$$h(x) = \frac{5}{(x+2)^2+4} = 5 \cdot f(x+2) \quad \text{donde } f(x) = \frac{1}{x^2+4} = \frac{1}{x^2+2^2}$$

$$\Rightarrow \hat{h}(\omega) = 5 \cdot e^{i2\omega}\, \hat{f}(\omega) = \frac{5\pi}{2}\, e^{2i\omega} \cdot e^{-2|\omega|}$$

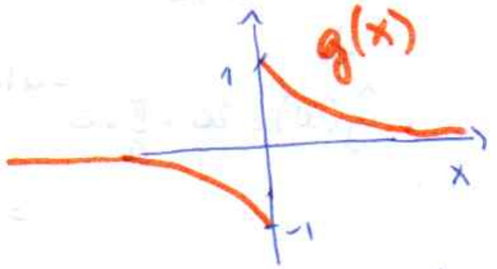
6). $g(x) = \underbrace{sg(x)}_{\substack{\text{sgn}(x) \\ \text{o sign}(x)}} \cdot e^{-|x|} = \begin{cases} -e^{-|x|} & \text{si } x<0 \\ e^{-|x|} & \text{si } x>0 \end{cases} = \begin{cases} -e^{x} & \text{si } x<0 \\ e^{-x} & \text{si } x>0 \end{cases}$

Observa: si $f(x) = e^{-|x|} = \begin{cases} e^{x} & \text{si } x<0 \\ e^{-x} & \text{si } x\geq 0 \end{cases}$

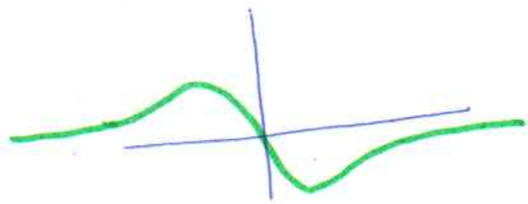
$$\Rightarrow f'(x) = \begin{cases} e^{x} & \text{si } x<0 \\ -e^{-x} & \text{si } x>0 \end{cases} = -g(x)$$

O sea: $g(x) = -f'(x)$.

$$\Rightarrow \hat{g}(\omega) = -\widehat{f'}(\omega) = -i\omega\, \hat{f}(\omega) = -i\omega \frac{2}{1+\omega^2} = \frac{-2i\omega}{1+\omega^2}$$



no continua



no es abs. integ.

Teorema de inversión

Sea $f \in L^1(\mathbb{R})$, continua por tramos, con f' continua por tramos. Entonces:

$$\frac{1}{2\pi} VP \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \text{ converge a } \frac{f(x^-)+f(x^+)}{2}$$

En particular,
Si f es continua en x:

$$\boxed{f(x) = \frac{1}{2\pi} VP \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega}$$

Si $\hat{f}(\omega) \in L^1(\mathbb{R})$:

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega}$$

Cambiando nombres: $x \leftrightarrow \omega$

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x) e^{ix\omega} dx$$

Llamando $g(x) = \hat{f}(x)$:

$$2\pi f(\omega) = \int_{-\infty}^{\infty} g(x) e^{-ix(-\omega)} dx = \hat{g}(-\omega)$$

Entonces: $\hat{\hat{f}}(-\omega) = 2\pi f(\omega)$

Ej: $f(x) = e^{-a|x|} \to \hat{f}(\omega) = \frac{2a}{a^2+\omega^2}$ $\quad g(x) = \frac{2a}{a^2+\cancel{x}^2}$

$$\downarrow$$

$$\hat{g}(\omega) = 2a \cdot \frac{\pi}{a} \cdot e^{-a|\omega|} = $$

$$= 2\pi f(-\omega)$$

(8 bis)

Ejemplo: Calcular $\int_{-\infty}^{\infty} \frac{\omega \operatorname{sen}(x\omega)}{1+\omega^2} d\omega$

Sabemos que por criterio Dirichlet-Abel, la integral converge.

Sea $f(x) = \begin{cases} e^{-x} & x>0 \\ 0 & x<0 \\ 1/2 & x=0 \end{cases}$

$$\hat{f}(\omega) = \frac{1}{1+i\omega} = \frac{1}{1+\omega^2} - i\frac{\omega}{1+\omega^2}$$

Teo inversión:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) \cdot e^{ix\omega} d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{1}{1+\omega^2} - i\frac{\omega}{1+\omega^2}\right)\left(\cos(x\omega) + i\operatorname{sen}(\omega x)\right) d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\cos(x\omega)}{1+\omega^2} + \frac{\omega \operatorname{sen}(\omega x)}{1+\omega^2}\right) d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\omega \operatorname{sen}(\omega x)}{1+\omega^2} = 2\pi f(x) - \int_{-\infty}^{\infty} \frac{\cos(x\omega)}{1+\omega^2} dx \underset{\text{si } x>0}{=} 2\pi e^{-x} - \pi e^{-x} = \boxed{\pi e^{-x}}$$

Sea $g(x) = e^{-|x|} \Rightarrow \hat{g}(\omega) = \frac{2}{1+\omega^2}$

Teo inversión: $g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\omega^2}\left(\cos(x\omega) + i\operatorname{sen}(\omega x)\right) d\omega$

$$e^{-|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(x\omega)}{1+\omega^2} d\omega$$

Si $x<0$, $f(x)=0$, $\int \frac{\omega \operatorname{sen}(\omega x)}{1+\omega^2} d\omega = 0 - \pi e^{x} = -\pi e^{x}$

Convolución

Sean $f, g \in L^1(\mathbb{R})$.
La convolución $f * g$ es:

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Si f y g son L1(R), y acotadas, esa integral converge absolutamente, y f*g es L1(R).

Teorema de convolución.

Sean $f, g \in L^1(\mathbb{R})$. Entonces:
y acotadas

$$\mathcal{F}(f*g)(\omega) = \mathcal{F}(f)(\omega) \cdot \mathcal{F}(g)(\omega).$$

o: $$\boxed{\widehat{f*g}(\omega) = \hat{f}(\omega) \cdot \hat{g}(\omega)}$$

Ejemplo. Hallar f tal que $\hat{f}(\omega) = \dfrac{1}{(1+\omega^2)^2}$.

Si $\hat{g}(\omega) = \dfrac{1}{1+\omega^2} \longrightarrow g(x) = \dfrac{e^{-|x|}}{2}$

$$\hat{f}(\omega) = \hat{g}(\omega) \cdot \hat{g}(\omega) \Rightarrow f(x) = (g*g)(x) = \int_{-\infty}^{\infty} g(x-t) \cdot g(t)\, dt$$

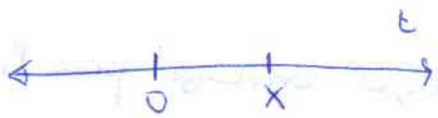
$$f(x) = \int_{-\infty}^{\infty} \frac{e^{-|x-t|}\, e^{-|t|}}{4}\, dt =$$

$$= \frac{1}{4}\left[\int_{-\infty}^{x} e^{t-x} \cdot e^{t}\, dt + \int_{x}^{0} e^{x-t} \cdot e^{t}\, dt + \int_{0}^{\infty} e^{x-t} \cdot e^{-t}\, dt\right]$$

si $x<0$

$$\frac{1}{4}\left[e^{-x} \cdot \frac{e^{2t}}{2}\Big|_{-\infty}^{x} - x \cdot e^{x} + e^{x} \cdot \frac{e^{-2t}}{-2}\Big|_{0}^{\infty}\right] = \frac{1}{4}\left[\frac{e^{x}}{2} - x e^{x} + \frac{e^{x}}{2}\right] = \frac{e^{x}}{4}(1-x)$$

si $x>0$:



$$= \frac{1}{4}\left[\int_{-\infty}^{0} e^{t-x} e^{t}\, dt + \int_{0}^{x} e^{t-x} \cdot e^{-t}\, dt + \int_{x}^{\infty} e^{x-t} \cdot e^{-t}\, dt\right]$$

$$= \frac{1}{4}\left[e^{-x} \frac{e^{2t}}{2}\Big|_{-\infty}^{0} + e^{-x} \cdot x + e^{x} \cdot \frac{e^{-2t}}{-2}\Big|_{x}^{\infty}\right] = \frac{1}{4}\left[\frac{e^{-x}}{2} + x e^{-x} + \frac{e^{-x}}{2}\right]$$

$$= \frac{e^{-x}}{4}(1+x)$$

si $x=0$

$$\frac{1}{4}\int_{-\infty}^{\infty} e^{-2|t|}\, dt = \frac{1}{2}\int_{0}^{\infty} e^{-2t}\, dt = \frac{1}{2} \cdot \frac{e^{-2t}}{(-2)}\Big|_{0}^{\infty} = \frac{1}{4}$$

$$f(x) = \begin{cases} \frac{1}{4} e^{x}(1-x) & x<0 \\ \frac{1}{4} e^{-x}(1+x) & x>0 \\ \frac{1}{4} & x=0 \end{cases} \Rightarrow f(x) = \frac{e^{-|x|}}{4}(1+|x|)$$

Formula de Plancherel

(o Formula de Parseval)

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega}$$

Ejemplo: Calcular $\int_{-\infty}^{\infty} \frac{\operatorname{sen}^2\omega}{\omega^2}\, d\omega$

Sabemos: $\hat{f}(\omega) = \frac{\operatorname{sen}\omega}{\omega}$ siendo $f(x) = \begin{cases} 1/2 & |x|<1 \\ 0 & |x|>1 \end{cases}$

Luego: $\int_{-1}^{1} \left(\frac{1}{2}\right)^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{\operatorname{sen}\omega}{\omega}\right|^2 d\omega$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{\operatorname{sen}^2\omega}{\omega^2}\, d\omega = 2\pi\int_{-1}^{1} \frac{1}{4}\, dx = 2\pi \cdot \frac{1}{4}\cdot 2 = \pi}$$

Ejemplo Calcular: $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\, dx$

Sabemos que si $f(x) = \operatorname{sg}(x)\, e^{-|x|} \Rightarrow \hat{f}(\omega) = \frac{-2i\omega}{1+\omega^2}$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\operatorname{sg}(x) e^{-|x|}\right)^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{-2i\omega}{1+\omega^2}\right|^2 d\omega$$

$$2\pi \cdot 2\cdot\int_{0}^{\infty} e^{-2x}\, dx = \int_{-\infty}^{\infty} 4\cdot\frac{\omega^2}{(1+\omega^2)^2}\, d\omega$$

$$\pi \cdot \frac{e^{-2x}}{-2}\Big|_0^{\infty} = \frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{\omega^2}{(1+\omega^2)^2}\, d\omega$$